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N-soliton solution for the derivative nonlinear Schrödinger equation with nonvanishing boundary conditions

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Abstract
For the case when all discrete spectral parameters are purely imaginary, an explicit N-soliton solution for the derivative nonlinear Schrödinger equation with nonvanishing boundary conditions, consisting of arbitrary number of pure bright and/or dark solitons, is derived. Shifts of soliton positions due to collisions between solitons are analytically obtained, which are irrespective of the bright or dark characters of the participating solitons. Typical collisions between solitons are graphically shown.

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1. Introduction

The derivative nonlinear Schrödinger equation (DNLSE) is a physically significant integrable model. It is a model describing nonlinear Alfvén waves in space plasma (see, e.g., [1–8]). It is equivalent to the modified nonlinear Schrödinger equation (MNLSE) under a gauge-like transform, which is one of the several integrable models describing sub-picosecond pulses in single-mode optical fibres (see, e.g., [9–13]). It was suggested that weak nonlinear electromagnetic waves in ferromagnetic [14], antiferromagnetic [15] or dielectric [16] systems under external magnetic fields can also be described by the DNLSE.

Solutions of the DNLSE under both the vanishing boundary conditions (VBC) and the nonvanishing boundary conditions (NVBC) are physically interesting topics. For problems of nonlinear Alfvén waves, weak nonlinear electromagnetic waves in magnetic and dielectric media, waves propagating strictly parallel to the ambient magnetic fields are modelled by the DNLSE with VBC while those oblique waves are modelled by the DNLSE with NVBC. For problems in optical fibres, pulses under bright background waves should be modelled by NVBC.

For the DNLSE with VBC, one-soliton solution was found by the inverse scattering transform (IST) [17], and N-soliton formulae were also derived by various approaches (see, e.g., [18, 19]). In this paper, the DNLSE specifically refers to the DNLSE considered in [17]. Other types of DNLSE have also been considered in the literature which are equivalent to this specific DNLSE under a gauge transformation [20–23].

For the DNLSE with NVBC, it has been shown that, in general, for a complex discrete spectral parameter, the one-soliton solution is a breather which degenerates to a pure bright soliton or a pure dark soliton when the discrete spectral parameter becomes purely imaginary [4, 24–26]. In known (1+1)-dimensional one-component integrable systems, the DNLSE with NVBC is a rare instance simultaneously admitting bright solitons, dark solitons, as well as their bound states (breathers). An explicit N-soliton solution describing interactions between these solitons is thus in special demand. However, like other NVBC problems, a double-valued function of the spectral parameter inevitably appears in the IST for the DNLSE with NVBC. An early IST performed on Riemann sheets [24] only gave an explicit expression for modulus of the one-soliton solution and asymptotic behaviours of the modulus of an implicit pure two-soliton solution [25]. Although the phase of the one-soliton solution was found later, yielding a very complicated solution [4, 5], it seems to be a tedious task to get an explicit multi-soliton solution based on the IST performed on Riemann sheets [24, 25]. Recently, by using the Bäcklund transformation, Steudel [27] got a N-soliton formula in terms of Vandermonde-like determinants, which is well suited for generating computer pictures but still unable to explicitly demonstrate collisions between solitons.

It has been suggested that constructing Riemann sheets for such NVBC problems can be avoided if one performs the IST on the plane of an appropriate affine parameter [28]. The technique was recently applied to the DNLSE with NVBC, yielding not only a much simpler one-soliton solution than those in the literature but also a simple IST for further research [26]. Immediately following [26], an infinite number of conservation laws were derived by a simple standard procedure [29] and the evolution of a rectangular initial pulse in the system was considered, which was shown to be highly nontrivial and significantly different from all known results [30].

However, the first Lax equation for the DNLSE, unlike those in usual integrable systems, is not an eigenequation of a linear operator, resulting in a potential-related phase η^+ in the Jost solutions. Only modulus of the soliton solution can be obtained directly from the IST [26]. One has to get η^+ from an integral relating η^+ with the modulus of the soliton solution (see (21)). For the N-soliton case, to find η^+ by directly integrating (21) is obviously impractical. We need to find another way to get η^+.

In this paper, we derive an explicit pure N-soliton solution for the DNLSE with NVBC, corresponding to purely imaginary discrete parameters. In section 2, we modify the IST derived in [26] to the case of purely imaginary discrete parameters. In section 3, we find a raw explicit N-soliton solution, leaving η^+ undetermined, by a technique similar to that in [31] which is very convenient in discussing asymptotic behaviours. In section 4, for the cases of $N = 1$ and $N = 2$, we find η^+ by directly integrating (21), yielding exact one-soliton and two-soliton solutions which show a common relation between η^+ and D, the denominator of the raw solution. The shift of soliton position due to collision is also obtained, which is irrespective of the bright or dark characters of the solitons. Collisions between solitons are graphically demonstrated. In section 5, we show that if the relation between η^+ and D can be generalized to the case of arbitrary N, the assumed solution demonstrates asymptotic behaviours similar to known integrable systems (see, e.g., [32]). It consists of N well-separated exact one-solitons at times long before ($t \to -\infty$) and long after ($t \to +\infty$) collisions. The total shift in position of any soliton due to multi-collisions is a simple summation of the shift due to each collision. These facts demonstrate that the assumed N-soliton solution appears to be an exact one.

2. Inverse scattering transform for the case of purely imaginary discrete spectral parameters

We write the DNLSE as

$$\mathrm{i}u_t + u_{xx} + \mathrm{i}(|u|^2 u)_x = 0, \tag{1}$$

where the subscript denotes the partial derivative. The first Lax equation is

$$\partial_x F = LF, \tag{2}$$

with

$$L = -\mathrm{i}\lambda^2\sigma_3 + \lambda U, \tag{3}$$

$$U = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}. \tag{4}$$

Here σ_i $(i = 1, 2, 3)$ are Pauli matrices, the bar stands for the complex conjugate and λ is the time-independent spectral parameter. One can find that (2), containing both λ and λ^2, is not a simple eigenequation of a linear operator.

As there is no phase shift across the DNLS solitons with NVBC [4, 26], the NVBC can be simply written as

$$u \to \rho, \qquad \text{as} \quad x \to \pm\infty, \tag{5}$$

where ρ is a positive constant. The asymptotic solutions of (2) are

$$E^{\pm}(x,k) = (I - \mathrm{i}\rho k^{-1}\sigma_1)\,\mathrm{e}^{-\mathrm{i}\lambda\zeta x\sigma_3}, \qquad \text{as} \quad x \to \pm\infty, \tag{6}$$

where

$$\lambda = \tfrac{1}{2}(k - \rho^2 k^{-1}), \qquad \zeta = \tfrac{1}{2}(k + \rho^2 k^{-1}), \tag{7}$$

and k is the affine parameter. We define Jost solutions,

$$\Psi(x,k) \to E^{+}(x,k), \qquad \text{as} \quad x \to \infty, \tag{8}$$

$$\Phi(x,k) \to E^{-}(x,k), \qquad \text{as} \quad x \to -\infty, \tag{9}$$

where

$$\Psi(x,k) = (\tilde{\psi}(x,k), \qquad \psi(x,k)), \tag{10}$$

$$\Phi(x,k) = (\phi(x,k), \qquad \tilde{\phi}(x,k)), \tag{11}$$

and the scattering coefficients by

$$\phi(x,k) = a(k)\tilde{\psi}(x,k) + b(k)\psi(x,k), \tag{12}$$

$$\tilde{\phi}(x,k) = \tilde{a}(k)\psi(x,k) - \tilde{b}(k)\tilde{\psi}(x,k). \tag{13}$$

$\psi(x,k), \phi(x,k)$ and $a(k)$ are analytic in the first and the third quadrants of the complex k plane, while $\tilde{\psi}(x,k), \tilde{\phi}(x,k)$ and $\tilde{a}(k)$ are analytic in the second and the fourth quadrants.

As shown in [26], on the plane of the affine parameter k, if $k_{n1} = k_n$ is a simple zero of $a(k)$ in the first quadrant, then $k_{n2} = -k_n$, $k_{n3} = \rho^2\bar{k}_n^{-1}$ and $k_{n4} = -\rho^2\bar{k}_n^{-1}$ are also simple zeros. At these zeros,

$$\phi(x, k_{nj}) = b_{nj}\psi(x, k_{nj}), \tag{14}$$

$$b_{n2} = -b_n, \qquad b_{n3} = \bar{b}_n, \qquad b_{n4} = -\bar{b}_n. \tag{15}$$

For the case when all discrete parameters λ_n $(n = 1, 2, \ldots, N)$ are purely imaginary, all zeros of $a(k)$ locate on the circle of radius ρ centred at the origin,

$$k_n = \rho \exp(\mathrm{i}\beta_n), \qquad 0 < \beta_n < \pi/2, \qquad n = 1, 2, \ldots, N. \tag{16}$$

That is, $k_{n3} = k_{n1}, k_{n4} = k_{n2}$. Actually there are only two zeros for each n. Contributions of k_{n3} and k_{n4} must be dropped from relevant equations obtained in [26]. For this case we also have $b_{n3} = b_{n1}$ and $b_{n4} = b_{n2}$. Then, $b_n = \bar{b}_n$, that is, b_n is real.

Therefore, for reflectionless potentials, dropping contributions of k_{n3} and k_{n4} in corresponding equations of [26], we get

$$a(k) = \exp\left(-\mathrm{i}2\sum_n \beta_n\right)\prod_{n=1}^{N}\frac{k^2 - k_n^2}{k^2 - \bar{k}_n^2}, \tag{17}$$

the inverse scattering equation,

$$\tilde{\psi}(x, k)\,\mathrm{e}^{\mathrm{i}\lambda\zeta x} = \begin{pmatrix}\mathrm{e}^{-\mathrm{i}\eta^+}\\ -\mathrm{i}\rho k^{-1}\,\mathrm{e}^{\mathrm{i}\eta^+}\end{pmatrix} + 2\sum_{n=1}^{N}\begin{pmatrix}k_n & 0\\ 0 & k\end{pmatrix}\frac{c_n\psi(x, k_n)}{k^2 - k_n^2}\,\mathrm{e}^{\mathrm{i}\lambda_n\zeta_n x}, \tag{18}$$

and the equation connecting soliton solution and the discrete Jost solutions,

$$u(x) = \rho\,\mathrm{e}^{-\mathrm{i}2\eta^+} - 2\rho\,\mathrm{e}^{-\mathrm{i}\eta^+}\sum_{n=1}^{N}\frac{c_n}{k_n}\psi_1(x, k_n)\,\mathrm{e}^{\mathrm{i}\lambda_n\zeta_n x}. \tag{19}$$

Here,

$$c_n = \frac{b_n}{\dot{a}(k_n)}, \qquad \lambda_n = \mathrm{i}\rho\sin\beta_n, \qquad \zeta_n = \rho\cos\beta_n, \tag{20}$$

and

$$\eta^+(x) = \frac{1}{2}\int_x^\infty (\rho^2 - |u|^2)\,\mathrm{d}x. \tag{21}$$

$a(k)$ is independent of t while [26]

$$b_n(t) = b_n(0)\,\mathrm{e}^{\mathrm{i}2\lambda_n\zeta_n(2\lambda_n^2-\rho^2)t} = b_n(0)\,\mathrm{e}^{\nu_n v_n t}, \tag{22}$$

where

$$\nu_n = \rho^2\sin(2\beta_n), \qquad v_n = \rho^2(1 + 2\sin^2\beta_n).$$

As $b_n(t)$ is real and

$$\dot{a}(k_n) = -\frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\beta_n}}{\rho\sin(2\beta_n)}\prod_{m\neq n}\frac{\sin(\beta_n - \beta_m)}{\sin(\beta_n + \beta_m)}, \tag{23}$$

we can set

$$c_n(t) = \mathrm{i}\chi_n\rho\sin(2\beta_n)\,\mathrm{e}^{\mathrm{i}\beta_n}\,\mathrm{e}^{\nu_n x_n}\,\mathrm{e}^{\nu_n v_n t}, \tag{24}$$

where x_n is a real constant and $\chi_n = \pm 1$. With a symmetric relation found in [26],

$$\tilde{\psi}(x, \rho^2 k^{-1}) = \mathrm{i}\rho^{-1}k\sigma_3\psi(x, k), \tag{25}$$

at $k = k_n$, we have

$$\tilde{\psi}(x, \bar{k}_n) = \mathrm{i}\rho^{-1}k_n\sigma_3\psi(x, k_n). \tag{26}$$

Therefore, at $k = \bar{k}_m$ $(m = 1, 2, \ldots, N)$, the first component of (18) is

$$\mathrm{i}\,\mathrm{e}^{\mathrm{i}\beta_m}\psi_1(x, k_m) = \mathrm{e}^{-\mathrm{i}\eta^+}\mathrm{e}^{\mathrm{i}\lambda_m\zeta_m x} + 2\sum_{n=1}^{N}\frac{k_n c_n\psi_1(x, k_n)}{\bar{k}_m^2 - k_n^2}\,\mathrm{e}^{\mathrm{i}(\lambda_m\zeta_m+\lambda_n\zeta_n)x}. \tag{27}$$

In principle, one can find $\psi_1(x, k_m)$ by solving linear equations (27), then get the N-soliton solution with (19) and (21).

3. A raw explicit N-soliton solution without determining η^+

Let

$$f_n = \sqrt{-\mathrm{i}c_n}\psi_1(x, k_n), \qquad g_n = \mathrm{e}^{-\mathrm{i}\beta_n}\sqrt{-\mathrm{i}c_n}\,\mathrm{e}^{\mathrm{i}\lambda_n\zeta_n x} \tag{28}$$

$$B_{nm} = \frac{g_n(2\rho\,\mathrm{e}^{\mathrm{i}2\beta_n})g_m}{k_n^2 - \bar{k}_m^2} = -\mathrm{i}g_n\frac{\mathrm{e}^{\mathrm{i}(\beta_n+\beta_m)}}{\rho\sin(\beta_n+\beta_m)}g_m. \tag{29}$$

(27) and (19) can be rewritten in matrix forms,

$$\boldsymbol{f} = -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\eta^+}\boldsymbol{g} - \boldsymbol{B}\boldsymbol{f}, \tag{30}$$

$$u = \rho\,\mathrm{e}^{-\mathrm{i}2\eta^+} - \mathrm{i}2\,\mathrm{e}^{-\mathrm{i}\eta^+}\boldsymbol{g}^T\boldsymbol{f}. \tag{31}$$

Here $\boldsymbol{B}_{nm} = B_{nm}$, $\boldsymbol{f}_n = f_n$. With (A.1), simple algebra yields a formal N-soliton solution,

$$u = \rho\,\mathrm{e}^{-\mathrm{i}2\eta^+}\frac{A}{D}, \tag{32}$$

where

$$\eta^+ = \frac{\rho^2}{2}\int_x^\infty \frac{|D|^2 - |A|^2}{|D|^2}\,\mathrm{d}x, \tag{33}$$

$$D = \det(\boldsymbol{I} + \boldsymbol{B}), \qquad A = \det(\boldsymbol{I} + \boldsymbol{B}'), \tag{34}$$

$$\boldsymbol{B}' = \boldsymbol{B} - 2\rho^{-1}\boldsymbol{g}\boldsymbol{g}^T, \tag{35}$$

$$B'_{nm} = B_{nm} - 2\rho^{-1}g_n g_m = \mathrm{e}^{-\mathrm{i}2(\beta_n+\beta_m)}B_{nm}. \tag{36}$$

Elements of $\boldsymbol{g}$, $\boldsymbol{B}$ and $\boldsymbol{B}'$ can be rewritten as

$$g_n = \sqrt{\chi_n}\rho^{\frac{1}{2}}\sqrt{\sin(2\beta_n)}\,\mathrm{e}^{-\mathrm{i}\frac{\beta_n}{2}}\,\mathrm{e}^{-\frac{\theta_n}{2}}, \tag{37}$$

$$B_{nm} = -\mathrm{i}\sqrt{\chi_n\chi_m}\frac{\sqrt{\sin(2\beta_n)\sin(2\beta_m)}}{\sin(\beta_n+\beta_m)}\,\mathrm{e}^{\mathrm{i}\frac{1}{2}(\beta_n+\beta_m)}\,\mathrm{e}^{-\frac{1}{2}(\theta_n+\theta_m)}, \tag{38}$$

$$B'_{nm} = -\mathrm{i}\sqrt{\chi_n\chi_m}\frac{\sqrt{\sin(2\beta_n)\sin(2\beta_m)}}{\sin(\beta_n+\beta_m)}\,\mathrm{e}^{-\mathrm{i}\frac{3}{2}(\beta_n+\beta_m)}\,\mathrm{e}^{-\frac{1}{2}(\theta_n+\theta_m)}, \tag{39}$$

in which

$$\theta_n = \nu_n(x - x_n - v_n t). \tag{40}$$

Actually, an N-soliton solution expressed in determinants is not convenient in discussing its asymptotic behaviour. Following mathematical techniques in [31], we can get a really explicit N-soliton solution by expanding D and A into series. Using (A.2), we have

$$\det(\boldsymbol{I} + \boldsymbol{B}) = 1 + \sum_{r=1}^{N}\sum_{1\leqslant n_1<n_2<\cdots<n_r\leqslant N} B(n_1, n_2, \ldots, n_r) \tag{41}$$

$$\det(\boldsymbol{I} + \boldsymbol{B}') = 1 + \sum_{r=1}^{N}\sum_{1\leqslant n_1<n_2<\cdots<n_r\leqslant N} B'(n_1, n_2, \ldots, n_r). \tag{42}$$

Here $B(n_1, n_2, \ldots, n_r)$ and $B'(n_1, n_2, \ldots, n_r)$ are rth-order principal minors of $\boldsymbol{B}$ and $\boldsymbol{B}'$, respectively. Further, with (A.3), we get

$$\begin{aligned} B(n_1, n_2, \ldots, n_r) &= (2\rho)^r \prod_n \mathrm{e}^{\mathrm{i}2\beta_n} f_n^2 \left(k_n^2 - \bar{k}_n^2\right)^{-1} \prod_{n<m} \left| \frac{k_n^2 - k_m^2}{k_n^2 - \bar{k}_m^2} \right|^2 \\ &= (-\mathrm{i})^r \,\mathrm{e}^{\mathrm{i}\sum_n \beta_n}\, \mathrm{e}^{-\sum_n \theta_n} \prod_n \chi_n \prod_{n<m} \frac{\sin^2(\beta_n - \beta_m)}{\sin^2(\beta_n + \beta_m)}, \end{aligned} \tag{43}$$

$$\begin{aligned} B'(n_1, n_2, \ldots, n_r) &= (2\rho)^r \prod_n \mathrm{e}^{-\mathrm{i}2\beta_n} f_n^2 \left(k_n^2 - \bar{k}_n^2\right)^{-1} \prod_{n<m} \left| \frac{k_n^2 - k_m^2}{k_n^2 - \bar{k}_m^2} \right|^2 \\ &= (-\mathrm{i})^r \,\mathrm{e}^{-\mathrm{i}3\sum_n \beta_n}\, \mathrm{e}^{-\sum_n \theta_n} \prod_n \chi_n \prod_{n<m} \frac{\sin^2(\beta_n - \beta_m)}{\sin^2(\beta_n + \beta_m)}, \end{aligned} \tag{44}$$

where $n, m \in \{n_1, n_2, \ldots, n_r\}$. So far we get a raw explicit N-soliton solution without explicitly determining η^+.

4. Exact one-soliton and two-soliton solutions

For the case of $N = 1$, we get

$$D = D_1 = 1 + B(1) = 1 - \mathrm{i}\chi_1 \,\mathrm{e}^{\mathrm{i}\beta_1}\, \mathrm{e}^{-\theta_1}, \tag{45}$$

$$A = A_1 = 1 + B'(1) = 1 - \mathrm{i}\chi_1 \,\mathrm{e}^{-\mathrm{i}3\beta_1}\, \mathrm{e}^{-\theta_1}. \tag{46}$$

Here $\theta_1 = \nu_1(x - x_1 - v_1 t)$. Directly integrating (21) yields

$$\eta^+ = \eta_1^+ = \mathrm{i}\ln \frac{D_1}{\bar{D}_1}, \tag{47}$$

and the exact one-soliton solution

$$u_1 = \rho \frac{A_1 D_1}{\bar{D}_1^2} = u_1(\theta_1), \tag{48}$$

which is identical to that obtained in the literature. It is a bright soliton for $\chi_1 = -1$ or a dark soliton for $\chi_1 = 1$. There is only one parameter, β_1, characterizing the soliton which is usually called the one-parameter soliton [5].

For the case of $N = 2$, we get

$$\begin{aligned} D = D_2 &= 1 + B(1) + B(2) + B(1,2) \\ &= 1 - \mathrm{i}\chi_1 \,\mathrm{e}^{\mathrm{i}\beta_1}\, \mathrm{e}^{-\theta_1} - \mathrm{i}\chi_2 \,\mathrm{e}^{\mathrm{i}\beta_2}\, \mathrm{e}^{-\theta_2} \\ &\quad - \chi_1\chi_2 \frac{\sin^2(\beta_1 - \beta_2)}{\sin^2(\beta_1 + \beta_2)} \,\mathrm{e}^{\mathrm{i}(\beta_1+\beta_2)}\, \mathrm{e}^{-\theta_1-\theta_2}, \end{aligned} \tag{49}$$

$$\begin{aligned} A = A_2 &= 1 + B'(1) + B'(2) + B'(1,2) \\ &= 1 - \mathrm{i}\chi_1 \,\mathrm{e}^{-\mathrm{i}3\beta_1}\, \mathrm{e}^{-\theta_1} - \mathrm{i}\chi_2 \,\mathrm{e}^{-\mathrm{i}3\beta_2}\, \mathrm{e}^{-\theta_2} \\ &\quad - \chi_1\chi_2 \frac{\sin^2(\beta_1 - \beta_2)}{\sin^2(\beta_1 + \beta_2)} \,\mathrm{e}^{-\mathrm{i}3(\beta_1+\beta_2)}\, \mathrm{e}^{-\theta_1-\theta_2}. \end{aligned} \tag{50}$$

Here $\theta_n = \nu_n(x - x_n - v_n t), n = 1, 2$. One can verify that

$$\mathrm{Re}(D_2)\frac{\mathrm{d}[\mathrm{Im}(D_2)]}{\mathrm{d}x} - \mathrm{Im}(D_2)\frac{\mathrm{d}[\mathrm{Re}(D_2)]}{\mathrm{d}x} = \frac{\rho^2}{4}(|D_2|^2 - |A_2|^2). \tag{51}$$

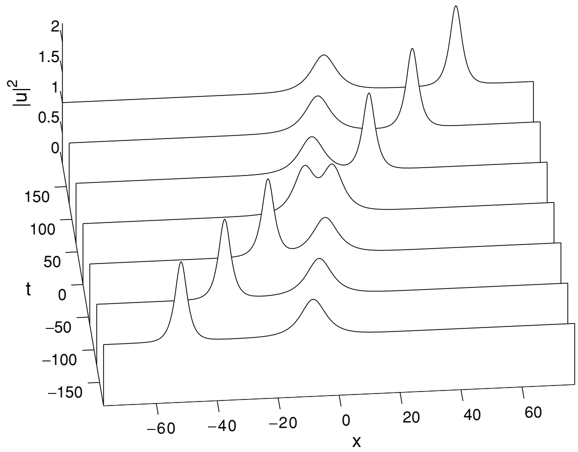

Figure 1. Collision between two bright solitons, $\rho = 1, \beta_1 = \pi/12, \beta_2 = \pi/24$. Variables in the figure are dimensionless.

With this relation, we get

$$\eta^+ = \eta_2^+ = \mathrm{i}\ln\frac{D_2}{\bar{D}_2}, \tag{52}$$

and the exact two-soliton solution,

$$u_2 = \rho\frac{A_2 D_2}{\bar{D}_2^2}. \tag{53}$$

Assume $\beta_2 > \beta_1$, that is, $v_2 > v_1$, at times long before collision ($t \to -\infty$), in the vicinity of $\theta_1 \approx 0, \theta_2 \to \infty, u_2 \approx u_1(\theta_1)$, while in the vicinity of $\theta_2 \approx 0, \theta_1 \to -\infty, u_2 \approx u_1(\theta_2 + \Delta)$, that is,

$$u_2 \approx u_1(\theta_1) + u_1(\theta_2 + \Delta), \tag{54}$$

in which

$$\Delta = 2\ln\left|\frac{\sin(\beta_1 + \beta_2)}{\sin(\beta_1 - \beta_2)}\right| > 0. \tag{55}$$

The solution consists of two well-separated solitons, moving to the positive direction of the x-axis, with the soliton of β_1 moving on the front.

At times long after collision ($t \to \infty$), in the vicinity of $\theta_1 \approx 0, \theta_2 \to -\infty, u_2 \approx u_1(\theta_1 + \Delta)$, while in the vicinity of $\theta_2 \approx 0, \theta_1 \to \infty, u_2 \approx u_1(\theta_2)$, that is,

$$u_2 \approx u_1(\theta_1 + \Delta) + u_1(\theta_2). \tag{56}$$

The solution consists of two well-separated solitons, moving to the positive direction of the x-axis, with the soliton of β_2 moving ahead.

These asymptotic behaviours imply that the soliton of β_2 overtakes the soliton of β_1, collides with the latter and emerges ahead of it. Owing to the collision, the fast soliton gets a forward shift $\Delta x_2 = \Delta/v_2$, while the slow soliton gets a backward shift $\Delta x_1 = -\Delta/v_1$, similar to solitons of other known integrable systems (see, e.g., [32]), regardless of whether they are bright or dark solitons.

Collisions between two solitons of typical parameters are shown in figures 1–4, where x_1 and x_2 are chosen as $x_1 = \Delta/(2v_1)$ and $x_2 = \Delta/(2v_2)$ so that the two solitons completely

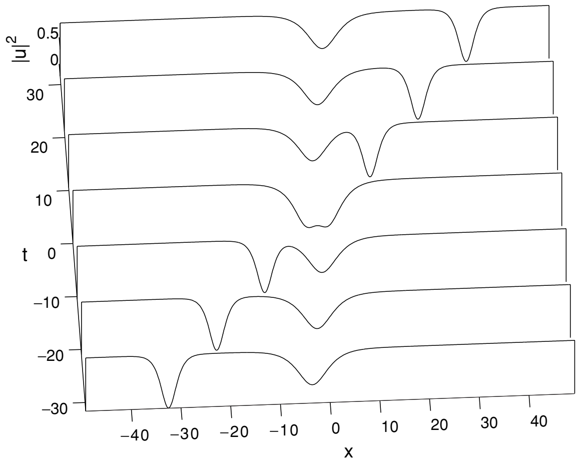

Figure 2. Collision between two dark solitons, $\rho = 1$, $\beta_1 = \pi/15$, $\beta_2 = \pi/6$. Variables in the figure are dimensionless.

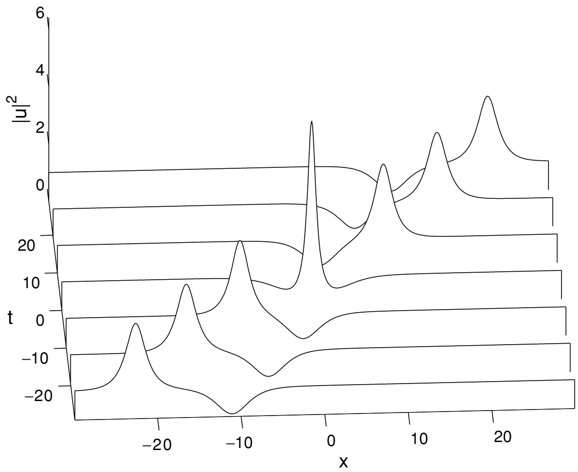

Figure 3. A bright soliton of $\beta_2 = 2\pi/15$ chases a dark soliton of $\beta_1 = 7\pi/60$, $\rho = 1$. Variables in the figure are dimensionless.

overlap at $t = 0$. For collisions between two bright solitons (figure 1) and two dark solitons (figure 2), two solitons overlap with two equal peaks at $t = 0$, and then interchange their roles without passing through each other. When a faster bright soliton chases a dark soliton (figure 3), the bright soliton is squeezed to a sharper peak while passing through the dark soliton. When a faster dark soliton chases a slower bright soliton (figure 4), the dark soliton simply passes through the bright soliton with little change in shape. Behaviours of solitons shown in figures 1–4 are in agreement with those in [25, 27] numerically obtained from an implicit two-soliton solution.

(47) and (52) prompt that

$$\eta^+(x) = \mathrm{i}\ln\frac{D}{\bar{D}}, \tag{57}$$

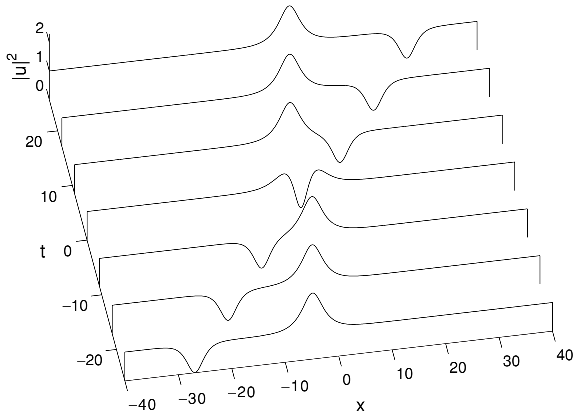

Figure 4. A dark soliton of $\beta_2 = \pi/6$ chases a bright soliton of $\beta_1 = \pi/12$, $\rho = 1$. Variables in the figure are dimensionless.

valid for one-soliton and two-soliton solutions, is possibly a general relation for arbitrary N, that is,

$$u = \rho \frac{AD}{\bar{D}^2} \tag{58}$$

is possibly the exact N-soliton solution. In the next section, an analysis of its asymptotic behaviour at large $|t|$ will show it appears to be true.

5. Asymptotic behaviour of the N-soliton solution

Assume (58) is valid for arbitrary N, let $\beta_1 < \beta_2 < \cdots < \beta_n$, i.e., $v_1 < v_2 < \cdots < v_n$, as $t \to -\infty$, in the vicinity of $\theta_n \approx 0$, we have

$$\theta_j \to \begin{cases} +\infty, & j > n, \\ -\infty & j < n, \end{cases} \tag{59}$$

$$D \approx B(1, 2, \ldots, n-1) + B(1, 2, \ldots, n-1, n), \tag{60}$$

$$A \approx B'(1, 2, \ldots, n-1) + B'(1, 2, \ldots, n-1, n). \tag{61}$$

With

$$B(1, 2, \ldots, n-1, n) = -\mathrm{i}\chi_n \,\mathrm{e}^{\mathrm{i}\beta_n - \theta_n} \prod_{j=1}^{n-1} \frac{\sin^2(\beta_j - \beta_n)}{\sin^2(\beta_j + \beta_n)} B(1, 2, \ldots, n-1), \tag{62}$$

and

$$B'(1, 2, \ldots, n-1, n) = -\mathrm{i}\chi_n \,\mathrm{e}^{-\mathrm{i}3\beta_n - \theta_n} \prod_{j=1}^{n-1} \frac{\sin^2(\beta_j - \beta_n)}{\sin^2(\beta_j + \beta_n)} B'(1, 2, \ldots, n-1), \tag{63}$$

in the vicinity of $\theta_n \approx 0$, we get

$$u \approx u_1(\theta_n + \Delta_n^-), \tag{64}$$

in which

$$\Delta_n^- = 2 \sum_{j=1}^{n-1} \ln \left| \frac{\sin(\beta_j + \beta_n)}{\sin(\beta_j - \beta_n)} \right|. \tag{65}$$

That is, the solution consists of N well-separated exact one-solitons,

$$u \approx \sum_{n=1}^{N} u_1(\theta_n + \Delta_n^-), \tag{66}$$

each characterized by one parameter β_n $(n = 1, 2, \ldots, N)$, moving to the positive direction of the x-axis, queueing up in a series of ascending β_n.

As $t \to \infty$, in the vicinity of $\theta_n \approx 0$, we have

$$\theta_j \to \begin{cases} -\infty, & j > n, \\ +\infty, & j < n, \end{cases} \tag{67}$$

$$D \approx B(n+1, n+2, \ldots, N) + B(n, n+1, \ldots, N), \tag{68}$$

$$A \approx B'(n+1, n+2, \ldots, N) + B'(n, n+1, \ldots, N). \tag{69}$$

With

$$B(n, n+1, \ldots, N) = -\mathrm{i}\chi_n\, \mathrm{e}^{\mathrm{i}\beta_n - \theta_n} \prod_{j=n+1}^{N} \frac{\sin^2(\beta_j - \beta_n)}{\sin^2(\beta_j + \beta_n)} B(n+1, n+2, \ldots, N), \tag{70}$$

and

$$B'(n, n+1, \ldots, N) = -\mathrm{i}\chi_n\, \mathrm{e}^{-\mathrm{i}3\beta_n}\, \mathrm{e}^{-\theta_n} \prod_{j=n+1}^{N} \frac{\sin^2(\beta_j - \beta_n)}{\sin^2(\beta_j + \beta_n)} B'(n+1, n+2, \ldots, N), \tag{71}$$

we also get, in the vicinity of θ_n,

$$u \approx u_1\big(\theta_n + \Delta_n^+\big), \tag{72}$$

in which

$$\Delta_n^+ = 2 \sum_{j=n+1}^{N} \ln\left|\frac{\sin(\beta_j + \beta_n)}{\sin(\beta_j - \beta_n)}\right|. \tag{73}$$

The solution also consists of N well-separated exact one-solitons,

$$u \approx \sum_{n=1}^{N} u_1\big(\theta_n + \Delta_n^+\big), \tag{74}$$

queueing up in a series of descending β_n. In the mean time the soliton of β_n overtakes the solitons of β_1 to β_n and is overtaken by the solitons of β_{n+1} to β_N. The total shift of its position is

$$\Delta x_n = \frac{1}{\nu_n}\big(\Delta_n^- - \Delta_n^+\big). \tag{75}$$

That is, due to collisions, the soliton of β_n got a total forward shift Δ_n^-/ν_n from exceeding those slower solitons of β_1 to β_{n-1} and got a total backward shift $\Delta_n^+\big/\nu_n$ from being exceeded by those faster solitons of β_{n+1} to β_N. Noting that Δ_n^- and Δ_n^+ are simply summations of shifts due to each collision between two solitons exactly obtained in the preceding section, we can conclude that the assumed N-soliton solution (58), with (34), (43) and (44), holds at least when no simultaneous collisions of more than two solitons occur. It appears to be an explicit N-soliton solution for the DNLSE with NVBC.

6. Summary and discussion

In this paper, an explicit pure N-soliton solution for the DNLSE with NVBC is derived for the case when all of the discrete spectral parameters are purely imaginary. The solution is explicitly written in a series of exponential functions. As an example of the solution, an explicit two-soliton solution is derived; typical collisions between two bright solitons, two dark solitons, as well as one bright soliton and one dark soliton, are graphically shown. Shifts of soliton positions due to collisions between solitons are analytically obtained. It is interesting to note that these shifts only depend on parameters of the participating solitons, irrespective of their bright or dark characters. In [25, 27], pure multi-soliton solutions were also considered, but no explicit closed form solution consisting of more than one soliton was obtained. When $\rho \to 0$, this pure N-soliton solution vanishes, that is, it has no counterpart in the regime of VBC. A general explicit N-soliton solution for complex discrete spectral parameters which can demonstrate collisions between breathers, as well as collisions between pure bright/dark solitons and breathers is still an open question. The approaches developed in this paper are promising to solve the problem. As it has been shown that, as $\rho \to 0$, the one-soliton solution for a complex discrete spectral parameter with NVBC approaches that with VBC [4, 26], we can expect that the general N-soliton solution with NVBC will also approach that with VBC.

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Appendix. Some useful formulae of linear algebra

If $\boldsymbol{b}$ and $\boldsymbol{g}$ are $N \times 1$ matrices, $\boldsymbol{A}$ is a regular $N \times N$ matrix, then

$$\boldsymbol{g}^T \boldsymbol{A}^{-1} \boldsymbol{b} = \frac{\det(\boldsymbol{A} + \boldsymbol{b}\boldsymbol{g}^T)}{\det(\boldsymbol{A})} - 1. \tag{A.1}$$

For a squared matrix $\boldsymbol{B}$

$$\det(\boldsymbol{I} + \boldsymbol{B}) = 1 + \sum_{r=1}^{N} \sum_{1 \leqslant n_1 < n_2 < \cdots < n_r \leqslant N} B(n_1, n_2, \ldots, n_r), \tag{A.2}$$

where $B(n_1, n_2, \ldots, n_r)$ is a rth-order principal minor of $\boldsymbol{B}$.

For a squared matrix $\boldsymbol{C}$ with elements $C_{jk+} = f_j g_k (x_j + y_k)^{-1}$,

$$\det(\boldsymbol{C}) = \prod_j f_j g_j \prod_{j<j'} (x_j - x_{j'}) \prod_{k<k'} (y_k - y_{k'}) \prod_{j,k} (x_j + y_k)^{-1}. \tag{A.3}$$

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